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VIDYASAGAR UNIVERSITY

Question Paper

B.Sc. Honours Examinations 2022

(Under CBCS Pattern)

Semester - IV

Subject : MATHEMATICS

Paper : C 9 - T

Multivariate Calculus

Full Marks : 60

Time : 3 Hours

Candidates are required to give their answers in their own words as far as practicable.
The figures in the margin indicate full marks.

1. Answer any ***five*** questions : 2×5=10

(a) Distinguish between double integral and repeated integral.

(b) Show that $\vec{\nabla}|\vec{r}|^n = n|\vec{r}|^{n-2}\,\vec{r}$, where $\vec{r} = xi + yj + zk$.

(c) Show that $\vec{\nabla}\Phi$ is a vector perpendicular to the surface $\Phi(x, y, z) = c$, where c is a constant.

(d) Write down the formula for the evaluation of length of a curve. Justify it.

(e) Show that $\lim\limits_{(x,y)\to 0} \dfrac{x-y^2}{x^2+y^2}$ does not exist.

P.T.O.

(f) Find the equation of the tangent plane to the surface $f(x, y) = x^2 + y^2 + \sin xy$ at the point (0, 2, 4).

(g) Find the surface area of a sphere by using surface of revolution.

(h) If $\vec{A}$ and $\vec{B}$ are irrotational, show that $\vec{A} \times \vec{B}$ is irrotational.

2. Answer any ***four*** questions : 5×4=20

(a) State and prove the Schwartz's theorem for the equality of f_{xy} and f_{yx} at some point (a, b) of the domain of definition of $f(x, y)$.

(b) Express $\int_0^{\frac{\pi}{2}} dx \int_0^{\cos x} x^2\, dy$ as a double integral and evaluate it.

(c) Prove $\vec{\nabla} \times (\vec{F} \times \vec{G}) = \vec{F}(\vec{\nabla}.\vec{G}) - \vec{F}.\vec{\nabla}\vec{G} + \vec{G}.\vec{\nabla}\vec{F} - \vec{G}(\vec{\nabla}.\vec{F})$, where $\vec{F}$ and $\vec{G}$ are differentiable vector function.

(d) Find $\iint_R f(x, y)\, dxdy$, over the region R bounded by $x = y^{\frac{1}{3}}$ and $x = \sqrt{y}$ where $f(x, y) = x^4 + y^2$.

(e) What is the maximum directional directional derivative of $g(x, y) = y^2 e^{2x}$ at (2, –1) and in the direction of what unit vector does it occur?

(f) Let f and g be twice differentiable functions of one variable and let $u(x, t) = f(x + ct) + g(x - ct)$ for a constant c. Show that $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$.

3. Answer any ***three*** questions : 10×3=30

(a) (i) Find the minimum value of $x^2 + y^2 + z^2$ subject to the constaint $ax + by + cz = 1 (a \neq 0, b \neq 0, c \neq 0)$.

(ii) Show that $f(x, y, z) = (x^2, y^2, z^2)^{-\frac{1}{2}}$ is harmonic. 8+2

P.T.O.

(b) (i) Let z be a differentiable function of x and y and let $x = r\cos\theta, y = r\sin\theta$,

Prove that $\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$. 7

(ii) Prove that $f(x,y) = \begin{cases} \frac{x^3+y^3}{x-y}, & x \neq y \\ 0, & x = y \end{cases}$ is not continuous at (0, 0). 3

(c) (i) Prove that $\iiint \frac{dxdydz}{x^2+y^2+(z-2)^2} = \pi\left(2 - \frac{3}{2}\log 3\right)$, extended over the sphere $x^2+y^2+z^2 \leq 1$.

(ii) Using a double integral, prove that the relation $B(m,n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}$,

m, n > 0. 5+5

(d) (i) Verify Stoke's theorem for the function $\vec{F} = x^2 i - xyj$ integrated round the square in the plane z = 0 and bounded by the lines x = 0, y = 0, $x = a$, $y = a$.

(ii) Prove that $\iint\left[2a^2 - 2a(x+y) - (x^2+y^2)\right] dxdy = 8\pi a^4$, the region of integration being the interior of the circle $x^2+y^2+2a(x+y) = 2a^2$. 6+4

(e) (i) Evaluate $\iint_S \overline{A}\,.\,\hat{n}\,ds$; $\overline{A} = 2yi - zj + x^2 k$ over the surface S of the bounded by the parabolic cylinder $y^2 = 8x$, in the first octant bounded by the plane y = 4 and z = 6. 7

(ii) Find the directional derivative of $f(x,y) = 2x^2 - xy + 5$ at (1, 1) in the direction of unit vector $\left(\frac{3}{5}, -\frac{4}{5}\right)$. 3

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VIDYASAGAR UNIVERSITY

B.Sc. Honours Examination 2021

(CBCS)

4th Semester

MATHEMATICS

PAPER—C9T

MULTIVARIATE CALCULUS

Full Marks : 60

Time : 3 Hours

The figures in the right-hand margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Answer any *four* questions. 4×15

1. (a) Let $f(x,y)=\begin{cases} x\sin\dfrac{1}{y}+y\sin\dfrac{1}{x}, xy\neq 0; \\ 0, xy=0. \end{cases}$

Show that at (0,0) the double limit exists but the repeated limits do not exist.

(b) Let $f(x,y)=\begin{cases}\dfrac{2xy}{x^2+y^2}, x^2+y^2\neq 0;\\ 0, x^2+y^2=0.\end{cases}$

Prove that f is a continuous function of either variable when the other variable is given a fixed value. Is f continuous at (0, 0)? Justify.

(c) If $u = f(x,y)$, where $x=r\cos\theta, y=r\sin\theta$; prove that

(i) $\left(\dfrac{\partial u}{\partial x}\right)^2+\left(\dfrac{\partial u}{\partial y}\right)^2=\left(\dfrac{\partial u}{\partial r}\right)^2+\dfrac{1}{r^2}\left(\dfrac{\partial u}{\partial \theta}\right)^2$;

(ii) $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}$. 4+4+7

2. (a) When is a function $f(x,y)$ said to be differentiable at a point (x,y)? State the sufficient condition for differentiability of (x,y).
Verify the sufficient condition for differentiability of the following function

$$f(x,y)=\begin{cases}x^2\sin\dfrac{1}{x}+y^2\sin\dfrac{1}{y}, x\neq 0, y\neq 0;\\ x^2\sin\dfrac{1}{x}, x\neq 0, y=0;\\ y^2\sin\dfrac{1}{y}, x=0, y\neq 0;\\ 0, x=0, y=0.\end{cases}$$

(b) Let $f(x,y)=\begin{cases}\dfrac{x^4+y^4}{x-y}, x\neq y;\\ 0, x=y\ .\end{cases}$

Show that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at (0,0). Examine the continuity of *f(x,y)* at (0, 0).

(c) If H be a homogeneous function in x and y of degree n having continuous first order partial derivatives and $u(x,y)=\left(x^2+y^2\right)^{-n/2}$, show that $\frac{\partial}{\partial x}\left(H\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(H\frac{\partial u}{\partial y}\right)=0$. 6+4+5

3. (a) Let $f(x,y)=\begin{cases}\frac{x^2y}{x^4+y^2},(x,y)\neq(0,0);\\ 0,(x,y)=(0,0).\end{cases}$

Show that f has a directional derivative at (0,0) in any direction $\beta=(l,m),\ l^2+m^2=1$, but f is discontinuous at (0, 0).

(b) If a function *f(x,y)* defined in a certain domain D of the xy-plane where $(a,b)\in D$ be such that both the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist in some neighbourhood of (a,b) and both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are differentiable at *(a,b)*, then prove that $f_{xy}(a,b) = f_{yx}(a,b)$.

(c) For the function $f(x,y)=\begin{cases}\frac{x^2y^2}{x^2+y^2},(x,y)\neq(0,0);\\ 0,(x,y)=(0,0);\end{cases}$

show that $f_{xy}(0,0)=f_{yx}(0,0)$. 4+6+5

4. (a) Prove that the volume of the greatest rectangular parallelepiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ *is* $\frac{8abc}{3\sqrt{3}}$.

(b) Find the stationary points of $f(x,y,z)=x^2y^2z^2$ subject to the condition $x^2+y^2+z^2=a^2$ (x, y, z are positive).

(c) Show that $\iiint (x+y+z)x^2y^2z^2 dx\ dy\ dz=\frac{1}{50400}$ taken throughout the tetrahedron bounded by three coordinate planes and x + y + z = 1.

5+4+6

5. (a) If E be the region bounded by the circle $x^2+y^2-2ax-2by=0$, show that

$$\iint_E \sqrt{x(2a-x)+y(2b-y)}\,dx\ dy=\frac{2\pi}{3}\left(a^2+b^2\right)^{\frac{3}{2}}.$$

(b) Prove that $\iiint_V \dfrac{dx\ dy\ dz}{x^2+y^2+\left(z-\frac{1}{2}\right)^2}=\pi\left(2+\frac{3}{2}\log 3\right)$

where V = $\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2\le 1\}$.

7+8

6. (a) In which direction from the point (1,3,2), the directional derivative of $\phi=2xz-y^2$ is maximum? What is the magnitude of this maximum?

(b) Is there a differentiable vector function $\vec{v}$ such that $curl\ \vec{v}=\vec{r}$? Justify it. Show that $\vec{E}=\frac{\vec{r}}{r^2}$ is irrotational. Find ϕ such that $\vec{E}=-\vec{\nabla}\phi$ and such that $\phi(a)=0$ where a > 0.

(c) If $\vec{A}=\left(4xy-3x^2z^2\right)\hat{i}+2x^2\hat{j}-2x^3z\hat{k}$, prove that $\int_C \vec{A}\bullet d\vec{r}$ is independent of the curve C joining two given points.

Is $\vec{A}\bullet d\vec{r}$ an exact differential? If yes, then solve the differential equation $\vec{A}\bullet d\vec{r}=0$.

3+6+6

7. (a) Prove $\nabla^2 r^n = n(n+1)r^{n-2}$, where n is a constant, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}, r = |\vec{r}|$.

(b) Prove that if $\int_{P_1}^{P_2} \vec{F} \bullet d\vec{r}$ is independent of the path joining any two points P_1 and P_2 in a given region, then $\oint \vec{F} \bullet d\vec{r} = 0$ for all closed paths in the region and conversely.

(c) Verify Green's theorem in the plane for $\oint_C \left(3x^2 - 8y^2\right) dx + (4y - 6xy)\, dy$, where C is the boundary of the region enclosed by : $y = \sqrt{x}, y = x^2$ 5+4+6

8. (a) Show that $\vec{F} = \left(2xy + z^3\right)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$ is a conservative force field. Find the scalar potential. Also evaluate the work done in moving an object in this field from (1,-2,1) to (3,1,4).

(b) Prove $\iint_S r^5 \hat{n}\, dS = \iiint_V 5r^3 \vec{r} dV$, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}, r = |\vec{r}|$.

(c) Evaluate by Stokes' theorem $\oint_C \sin z\ dx - \cos x\ dy + \sin y\ dz$, where C is the boundary of the rectangle : $0 \le x \le \pi, 0 \le y \le 1, z = 3$. 6+5+4

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2019

B.Sc. (Honours)

4th Semester Examination

MATHEMATICS

Paper - C9T

(Multivariate Calculus)

Full Marks : 60 Time : 3 Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable. Illustrate the answers wherever necessary.

Unit - I

1. Answer any ***three*** questions : 2×3

(a) Show that the limit exists at the origin but the repeated limit does not, for the function

$$f(x, y) = \begin{cases} x\sin\dfrac{1}{y} + y\sin\dfrac{1}{x}, & xy \neq 0 \\ 0, & xy = 0 \end{cases}$$

[Turn Over]

(b) For $F(x, y) = x^4 y^2 \sin^{-1}\frac{y}{x}$ show that

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} = 6F$$

(c) Define directional derivative of the function $f(x, y)$ at the point (a, b). Obtain partial derivative as a special case of it.

(d) Is $f(x, y) = |y|(1+x)$ differentiate at (0, 0) ?

(e) Find the maximum or minimum value of

$f(x, y) = x^3 + y^3 - 3axy.$

2. Answer any ***one*** question : 5×1

(a) State and prove sufficient condition for differentiability of a function $f(x, y)$ at a point (a, b).

(b) Let $(a, b) \in D$, the domain of definition of f. If $f_x(a, b)$ exist and $f_y(x, y)$ is continuous at (a, b) then show that $f(x, y)$ is differentiable at (a, b).

3. Answer any ***one*** question 10×1

(a) (i) Find the shortest distance from the origin to the hyperbola $x^2 + 8xy + 7y^2 = 225, z = 0$.

(ii) If z be a differentiable function of x and y and if $x = c\cosh(u)\cos(v)$, $y = c\ \sin hv$ $\sin v$ then prove that

$$\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}=\frac{1}{2}c^2\ (\cosh 2u - \cos 2v)$$

$$\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right)$$ 5+5

(b) (i) Define total differential of a function $f(x, y, z)$.

Approximate the change in the hypotenuse of a right angled triangle whose sides are 6 and 8 cm, when the shorter side is lengthened by $\left(\frac{1}{4}\text{cm}\right)$ and the longer is shortened by $\left(\frac{1}{8}\text{cm}\right)$.

[*Turn Over*]

(ii) Prove that the volume of the greatest rectangular parallelopiped, that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, is $\frac{8abc}{3\sqrt{3}}$. (2+3)+5

Unit - II

4. Answer any ***two*** questions : 2×2

(a) Let

$$f(x, y) = \begin{cases} \frac{1}{2}, & y = \text{rational} \\ x, & y = \text{irrational} \end{cases}$$

verify whether $\int_0^1 dy \int_0^1 f(x, y)dx$ exists or not.

(b) Evaluate $\int_0^\infty \frac{\sin rx}{x} dx$ from $\int_0^\infty \int_0^\infty e^{-xy} \sin rx \, dx \, dy$ with the help of change of order of integration.

(c) Evaluate $\iint\limits_R (x^2+y^2)\,dx\,dy$ over the region R bounded by $xy=1$, $y=0$, $y=x$, $x=2$.

5. Answer any ***two*** questions : 5×2

(a) Show in a diagram the field of integration of the integral $\int_0^1 \left(\int_x^{1/x} \frac{y\,dy}{(1+xy)^2(1+y^2)} \right) dx$ and by changing the order of integration, show that the value of the integral is $\frac{\pi-1}{4}$.

(b) Are the two iterated integrals $\int_1^{\infty} dx \int_1^{\infty} \frac{x-y}{(x+y)^3}\,dy$ and $\int_1^{\infty} dy \int_1^{\infty} \frac{x-y}{(x+y)^3}\,dx$ equal? Justify your answer.

(c) Evaluate

$$\iiint\limits_E \sqrt{a^2b^2c^2 - b^2c^2x^2 - a^2c^2y^2 - a^2b^2z^2}\; dx\,dy\,dz$$

[Turn Over]

where E is the region bounded by the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Unit - III

6. Answer any *three* questions : 2×3

(Symbols have their usual meaning)

(a) Find the total work done in moving a particle in a force field given by

$$F=(2x-y+z)\hat{i}+(x+y-z)\hat{j}+(3x-2y-5z)\hat{k},$$

along a circle C in the xy-plane $x^2+y^2=9$, $z=0$.

(b) Evaluate the vector line integral $\int_C \vec{F}\times d\vec{x}$ where

$\vec{F}=Z\hat{i}$ and C is the part of the circular helix $\vec{x}=b\cos t\hat{i}+b\sin t\hat{j}+c\,t\hat{k}$ between the points $(-b, 0, \pi c)$ and $(b, 0, 0)$.

(c) Prove that $\vec{\nabla}.\left[r\vec{\nabla}\left(\frac{1}{r^3}\right)\right]=3r-4$, where $\vec{r}$ is the position vector and $r=|\vec{r}|$

(d) Find the equation of the tangent plane to the surface $xyz = 4$ at the point (1, 2, 2).

(e) If $\Delta\phi = (2xyz^3, x^2z^3, 3x^2yz^2)$ and

$\phi(1,-2,2) = 4$, find the function ϕ.

7. Answer any ***one*** question : 10×1

(a) (i) If $\vec{\nabla}.\vec{E} = 0, \vec{\nabla}.\vec{H} = 0, \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{H}}{\partial t}$ and

$\vec{\nabla}\times\vec{H} = \frac{\partial\vec{E}}{\partial t}$, then show that

$$\nabla^2\vec{H} = \frac{\partial^2\vec{H}}{\partial t^2} \text{ and } \nabla^2\vec{\mathrm{E}} = \frac{\partial^2\vec{\mathrm{E}}}{\partial \mathrm{t}^2}.$$

(ii) Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k},\ r = |\vec{r}\,|$ and $f(x)$ is a scalar function possessing first and 2nd order derivatives prove that

$$\nabla^2 f(x) = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}.$$

[*Turn Over*]

If $\nabla^2 f(r) = 0$, show that $f(r) = A + \frac{B}{r}$

where A and B are arbitrary constants.

(b) (i) Prove that

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla}.\vec{A}) - \vec{\nabla}^2 \vec{A}$$

(ii) Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}|$.

If $f(r) = \log r$ and $g(r) = 1/r$, $r \neq 0$.

Satisfy $2\vec{\nabla}f + h(r)\vec{\nabla}g = 0$ then find $h(r)$.

Unit - IV

8. Answer any *two* questions : 2×2

(a) Evaluate

$$\int_S (x^2 dy\, dz + y^2 z\, dz\, dx + 2z\,(xy - x - y)\, dx\, dy)$$

where S is the surface of the cube

$$0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$$

(b) Show that $\iint_S \vec{r}.\vec{ds} = 3v$ where v is the volume enclosed by the closed surface S and $\vec{r}$ has its usual meaning.

(c) (i) State Green's theorem in the plane.

(ii) If S be any closed surface enclosing a volume V and $\vec{F} = x\hat{i} + 2y\hat{j} + 3z\hat{k}$, prove that $\iint\limits_S \vec{F}.\hat{n}\, ds = 6V$.

9. Answer any *one* question : 5×1

(a) Evaluate $\iint\limits_S \vec{F}.\hat{n}\, dS$, where

$\vec{F} = x\hat{i} - y\hat{j} + (z^2 - 1)\hat{k}$, S is the surface of the region bounded by $x^2 + y^2 = 4$, $z = 0$, $z = 4$ in the first octant.

(b) Verify Green's theorem in the plane for $\oint (xy + y^2)dx + x^2 dy$ where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.